

Approximation of Biased Boolean Functions of Small Total Influence by DNF's

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Abstract

The influence of the k 'th coordinate on a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is the probability that flipping x_k changes the value $f(x)$. The total influence $I(f)$ is the sum of influences of the coordinates. The well-known ‘ Junta Theorem’ of Friedgut (1998) asserts that if $I(f) \leq M$, then f can be ϵ -approximated by a function that depends on $O(2^{M/\epsilon})$ coordinates. Friedgut’s theorem has a wide variety of applications in mathematics and theoretical computer science.

For a biased function with $\mathbb{E}[f] = \mu$, the edge isoperimetric inequality on the cube implies that $I(f) \geq 2\mu \log(1/\mu)$. Kahn and Kalai (2006) asked, in the spirit of the Junta theorem, whether any f such that $I(f)$ is within a constant factor of the minimum, can be $\epsilon\mu$ -approximated by a DNF of a ‘small’ size (i.e., a union of a small number of sub-cubes). We answer the question by proving the following structure theorem: If $I(f) \leq 2\mu(\log(1/\mu) + M)$, then f can be $\epsilon\mu$ -approximated by a DNF of size $2^{2^{O(M/\epsilon)}}$. The dependence on M is sharp up to the constant factor in the double exponent.

1 Introduction

1.1 Background

Let f be a Boolean function on the discrete cube, that is, $f : \{0, 1\}^n \rightarrow \{0, 1\}$. The *influence* of the k 'th coordinate on f is

$$I_k(f) = \Pr[f(x) \neq f(x \oplus e_k)],$$

where $x \oplus e_k$ is obtained from x by flipping the k 'th coordinate and leaving the other coordinates unchanged. The *total influence* (or, in short, the *influence*) of f is defined as $I(f) = \sum_{k=1}^n I_k(f)$.

The notion of influences appears naturally in many contexts, such as isoperimetric inequalities (as $I(f)$ equals, up to normalization, to the *edge boundary* of the subset $\{x : f(x) = 1\}$ of the discrete cube), threshold phenomena in random graphs, cryptographic properties of election functions, etc. As a result, the last three decades witnessed a very extensive study of the ‘theory of influences’, that has led to numerous applications in areas as diverse as theoretical

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computer science (e.g., hardness of approximation [7, 17] and machine learning [27]), percolation theory [2], social choice theory [25], and others (see the survey [21]).

The minimal possible value of the total influence, as function of the expectation $\mathbb{E}[f]$, can be derived from the classical *edge isoperimetric inequality on the cube* [1, 15, 16, 24], which asserts that for any m , among all the m -element subsets of the discrete cube, the minimal edge boundary is attained by the set of the m largest elements in the lexicographic order. A weaker (but more convenient and so more widely-used) bound is:

Theorem 1.1 (Harper, Bernstein, Lindsey, Hart). *For any $f : \{0, 1\}^n \rightarrow \{0, 1\}$, we have*

$$I(f) \geq 2\mu(f) \log(1/\mu(f)),$$

where $\mu(f) := \mathbb{E}[f]$. Equality is attained if and only if f is a sub-cube.

One of the best-known and most widely-used results on influences is Friedgut’s ‘Junta Theorem’ [11] which describes the structure of functions with a low influence:

Theorem 1.2 (Friedgut). *Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a balanced Boolean function (i.e., $\mathbb{E}[f] = 1/2$) that satisfies $I(f) \leq M$, and let $\epsilon > 0$. Then there exists a Boolean function g that ϵ -approximates f (i.e., $\Pr[f(x) \neq g(x)] \leq \epsilon$) such that g depends on $2^{O(M/\epsilon)}$ coordinates. The dependence on M is sharp, up to a multiplicative constant.*

For a balanced function f , Theorem 1.1 implies that $I(f) \geq 1$. Hence, Theorem 1.2 may be viewed as a *structure theorem* for balanced functions with influence within a constant multiplicative factor of the minimum possible.

While balanced functions and the uniform measure on the discrete cube are sufficient for many of the applications of Theorem 1.2, some applications – most notably, to threshold phenomena in random graphs and other structures – require to generalize the results to biased functions (i.e., $\mathbb{E}[f] \neq 1/2$), and to the setting of the *biased measure* μ_p on the discrete cube, defined by $\mu_p(x) = p^{\sum x_i} (1-p)^{n-\sum x_i}$. Theorem 1.2 extends easily to these settings. However, the dependence of the results on $\mathbb{E}[f]$ (resp. on p) is such that they become much less informative when $\mathbb{E}[f] = o(1)$ or $\mathbb{E}[f] = 1 - o(1)$ (resp. $p = o(1)$ or $p = 1 - o(1)$), as the size of the approximating Junta g becomes ‘too large’.

The case of balanced functions with respect to a biased measure was studied in numerous works and led to breakthrough results on the sharpness of thresholds of graph properties, such as the k -SAT problem (see Friedgut [12], Bourgain [4], Bourgain-Kalai [5], and Hatami [18]). In a nutshell, it was shown that while influence within a constant factor of the minimum possible does not imply that the function can be approximated by a Junta, it allows to say that the function admits a weaker structure called in [18] ‘pseudo-Junta’, and if it is ‘somewhat symmetric’ then stronger structural properties hold [5, 12].

The case of biased functions with a *very low* influence was also studied in a number of works. Those works aimed at proving a *stability version* of the edge isoperimetric inequality on the cube, asserting that if the influence of f is within a small (additive) distance of the minimum possible, then f is close (in the ℓ^1 norm) to the indicator function of an extremal family. After a series of works which proved stability in specific cases (Friedgut, Kalai and Naor [14], Bollobás, Leader and Riordan (unpublished), Samorodnitsky [29], and Ellis [8]), the authors and Ellis recently proved stability for all values of $\mathbb{E}[f]$, obtaining the following structure theorem, which is sharp up to an absolute constant factor.

Theorem 1.3 ([9]). *Let $\epsilon > 0$ and let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function with $\mathbb{E}[f] = \mu$, such that $I(f) \leq I(\mathcal{L}_\mu) + \epsilon$, where \mathcal{L}_μ is the characteristic function of the set of the $\mu 2^n$ maximal elements in the lexicographic order. Then there exists a Boolean function g such that $\{x : g(x) = 1\}$ is ‘weakly isomorphic’ to \mathcal{L}_μ , and $\Pr[f(x) \neq g(x)] \leq C\epsilon$, where C is a universal constant. The result is sharp up to the value of the constant C .*

While Theorem 1.3 solves the ‘stability’ question (up to an absolute constant factor), it does not tell anything about the structure of functions whose influence is larger than the minimum by $\Omega(\mu)$, let alone functions whose influence is within a constant multiplicative factor of the minimum.

1.2 The structure of low-influence biased functions

In [19], motivated by the study of threshold phenomena in random graphs and hypergraphs, Kahn and Kalai suggested to study the structure of biased Boolean functions whose influence lies within a constant factor of the minimum possible, i.e., $I(f) \leq C\mu \log(1/\mu)$, where $\mu := \mathbb{E}[f]$. It is clear that such functions cannot be approximated by a constant-size Junta (as even the sub-cube of measure μ , whose influence is the minimum possible, cannot be approximated by a function that depends on less than $\log(1/\mu)$ coordinates). Instead, the authors of [19] conjectured that f can be approximated by a DNF of a small width.

Conjecture 1.4 (Kahn and Kalai). *For any $C, \epsilon, \mu > 0$, there exists $w = O_{C, \epsilon}(\log(1/\mu))$ such that the following holds. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a monotone Boolean function with $\mathbb{E}[f] = \mu$. Suppose that $I[f] \leq C\mu \log(1/\mu)$. Then f can be $\epsilon\mu$ -approximated by a DNF of width at most w (i.e., a union of sub-cubes of co-dimension at most w).*

It should be noted that the natural adaptation of Theorem 1.2 to the setting of Kahn-Kalai yields the following:

Theorem 1.5 (Friedgut). *For any $C, \epsilon, \mu > 0$, there exists $j = j(C, \epsilon, \mu) = (1/\mu)^{O(C/\epsilon)}$ such that the following holds. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function such that $\mathbb{E}[f] = \mu$. Suppose that $I(f) \leq C\mu \log(1/\mu)$. Then f can be $\epsilon\mu$ -approximated by a j -Junta (i.e., a function that depends on at most j coordinates).*

This result, which is tight up to the constant in the exponent, does not tell anything when μ is polynomial in n^{-1} , as is the case for many applications. Kahn and Kalai hoped that by replacing the ‘Junta approximation’ with approximation by a DNF, one can obtain a meaningful structure result also for polynomially small μ .

1.3 Our results

Unfortunately, as we show below, Conjecture 1.4 is too strong, and in fact, the width of the best approximating DNF may be as large as $2^{O_{C, \epsilon}(\log(1/\mu))}$, which (like Theorem 1.5) tells us nothing for μ polynomially small in n . On the other hand, we show that (a variant of) Conjecture 1.4 does hold if the assumption on $I(f)$ is a bit stronger. Our main result is the following:

Theorem 1.6. *For any $M, \epsilon > 0$, there exists $s = s(M, \epsilon) = 2^{2^{O(M/\epsilon)}}$ such that the following holds. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function such that $\mathbb{E}[f] = \mu$. Suppose that $I(f) \leq 2\mu(\log(1/\mu) + M)$. Then f can be $\epsilon\mu$ -approximated by a DNF of size s (i.e., a union of s sub-cubes). Consequently, f can be $\epsilon\mu$ -approximated by a DNF of width at most $\log(1/\mu) + 2^{O(M/\epsilon)}$ (i.e., a union of sub-cubes of co-dimension at most $\log(1/\mu) + 2^{O(M/\epsilon)}$).*

Sharpness of the result. Theorem 1.6 is sharp, up to the constant in the exponent. The sharpness example is the intersection of a sub-cube of co-dimension $\approx \log(1/\mu)$ with the *dual tribes function* introduced by Ben-Or and Linial [3].

For $w, s \in \mathbb{N}$, the *tribes* function $\text{Tribes}_{w,s} : \{0, 1\}^{ws} \rightarrow \{0, 1\}$ is defined as

$$\text{Tribes}_{w,s}(x_1, \dots, x_{ws}) = (x_1 \wedge \dots \wedge x_w) \vee (x_{w+1} \wedge \dots \wedge x_{2w}) \vee \dots \vee (x_{(s-1)w+1} \wedge \dots \wedge x_{sw}), \quad (1)$$

and the *dual tribes* function $\text{Tribes}_{w,s}^\dagger : \{0, 1\}^{ws} \rightarrow \{0, 1\}$ is defined as

$$\text{Tribes}_{w,s}^\dagger(x_1, \dots, x_{ws}) = 1 - \text{Tribes}_{w,s}(1 - x_1, \dots, 1 - x_{ws}). \quad (2)$$

Now, let $w, l \in \mathbb{N}$, let $n = w2^w + l$, and let f be the function

$$f(\mathbf{x}) = \begin{cases} \text{Tribes}_{w,2^w}^\dagger(x_1, x_2, \dots, x_{n-l}) & x_{n-l+1} = \dots = x_n = 1 \\ 0 & \text{Otherwise} \end{cases}.$$

Write $\mu = \mathbb{E}[f]$. As we show in Section 5, we have $I(f) = 2\mu(\log(1/\mu) + \Theta(w))$, but f cannot be 0.2μ -approximated by any DNF of width at most $\log \frac{1}{\mu} + \Theta(2^w)$. In addition, f cannot be 0.1μ -approximated by a DNF of size at most $2^{\Theta(2^w)}$. This shows the sharpness of Theorem 1.6, and also provides a counterexample for Conjecture 1.4.

Range of applicability and meaning of the result. Theorem 1.6 is ‘interesting’ in the range

$$2\mu(\log(1/\mu) + \Omega(\mu)) \leq I(f) \leq 2\mu(\log(1/\mu) + o(\log(1/\mu))). \quad (3)$$

For values of the influence smaller than the l.h.s. of (3), Theorem 1.3 can be applied to get approximation by a single sub-cube. For values larger than the r.h.s. of (3), i.e., $I(f) \geq c\mu \log(1/\mu)$ for $c > 2$, a stronger assertion can be deduced from the Junta approximation of Friedgut’s Theorem 1.5.

For $I(f)$ in the range (3), on the one hand, one cannot hope for approximation by a single sub-cube, as it can be easily seen that the union of s sub-cubes satisfies $I(f) = 2\mu(\log(1/\mu) + \Theta_s(\mu))$. On the other hand, the best one can obtain using Theorem 1.5 is approximation by a Junta of size $\Omega(1/\mu)$. Our Theorem 1.6 provides approximation by a DNF whose size is much smaller, and in particular, by a *constant-size DNF* for any constant M . Hence, it seems to be the ‘right’ structure result one would like to achieve, at least in the range $I(f) = 2\mu(\log(1/\mu) + \Theta(1))$.

Our techniques. Like the proof of Friedgut’s Junta theorem, our proof makes use of discrete Fourier analysis and hypercontractivity, via the seminal KKL theorem [20]. In addition, we use the classical *combinatorial shifting* technique [6, 10]. To be more specific, the central novel ingredient in our proof is the following lemma, that (along with its proof method) may be of independent interest.

Lemma 1.7. *There exists an absolute constant C_1 such that the following holds. Let $M, \delta > 0$ satisfy $M/\delta > C$, and let $\mu \in (0, 1 - \delta)$. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function with $\mathbb{E}[f] = \mu$, and suppose that $I(f) \leq 2\mu(\log(1/\mu) + M)$. Then*

$$\max_{i \in [n]} \{I_i(f)\} \geq 2^{-C_1 M/\delta} \mu.$$

Lemma 1.7 asserts that if the total influence of f is ‘small’, then f must have an influential coordinate. For μ bounded away from 0 and 1, the Lemma follows immediately from the KKL

theorem. We leverage the result to any measure μ by an inductive argument, based on the shifting technique.

Organization of the paper. In Section 2 we introduce notations to be used throughout the paper and describe the general structure of the proof of Theorem 1.6. In Section 3 we prove the main lemmas we use in the sequel, including Lemma 1.7. The proof of Theorem 1.6 is presented in Section 4. The sharpness examples are presented in Section 5, and we conclude the paper with a few open problems in Section 6.

Note. Keevash and Long [22] have independently and simultaneously proved another version of our main theorem, with an upper bound of $2^{2^{O(M/\epsilon)^2}}$ on the size of the DNF (instead of our sharp $2^{2^{O(M/\epsilon)}}$). The methods of [22] is different from ours. Essentially, while we obtain our main lemma (i.e., Lemma 1.7 which asserts the existence of an influential coordinate) using combinatorial shifting and the classical KKL theorem, in [22] a slightly weaker version of the main lemma is obtained using ‘heavier’ analytic tools, including inequalities of Talagrand and Polyanskiy.

2 Notations and Proof Overview

2.1 Notations

First, for sake of completeness we give the formal definition of a DNF and its width and size.

A *literal* is either a variable x_i or its negation. A *term* is an AND of literals, and a *DNF* is an OR of terms. E.g., the following $(x_1 \wedge \neg x_2) \vee (x_2 \wedge x_3) \vee (x_3 \wedge \neg x_4 \wedge x_5)$ is a DNF formula. Let $D = T_1 \vee T_2 \vee \dots \vee T_s$ be a DNF. The *size* of D is the amount of literals in D (i.e., s). The *width* of D is the maximal number of literals in a term of D . (So, the above DNF has size 3 and width 3). We identify a DNF on n variables with the Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ defined as $f(x_1, \dots, x_n) = 1$ if and only if (x_1, \dots, x_n) satisfies the formula. Note that each term corresponds to a subcube, a DNF of size s corresponds to the characteristic function of the union of s subcubes, and its width is the maximal co-dimension of a sub-cube that corresponds to one of its terms.

Throughout the paper, $[n]$ denotes the set $\{1, 2, \dots, n\}$, and C, c, C_i denote universal constants. f will be denote a Boolean function, i.e., $f : \{0, 1\}^n \rightarrow \{0, 1\}$, and $\mathbb{E}[f]$ will be denoted by $\mu(f)$ or simply by μ . We will assume throughout that $I_1(f)$ is the maximal influence of f . (There is no loss of generality in this assumption, as we can always reorder the coordinates of f .) We let $s_f(\epsilon)$ be the minimal size of a DNF that $\epsilon\mu(f)$ -approximates f , and define M to be such that

$$I(f) = 2\mu(f)(\log(1/\mu(f)) + M).$$

(Note that $M \geq 0$ by Theorem 1.1.)

The proof of Theorem 1.6 will use an inductive approach, for which we will persistently use the following notations. For a function f , we let $f_1, f_0 : \{0, 1\}^{n-1} \rightarrow \{0, 1\}$ be the Boolean functions defined by

$$f_1(x_1, \dots, x_{n-1}) = f(1, x_1, \dots, x_{n-1}), \quad f_0(x_1, \dots, x_{n-1}) = f(0, x_1, \dots, x_{n-1}).$$

We write $\mu_1 = \mu(f_1)$ and $\mu_0 = \mu(f_0)$. Similarly, we let $M_1, M_0 \geq 0$ be the numbers satisfying

$$I(f_1) = 2\mu_1 \left(\log \frac{1}{\mu_1} + M_1 \right), \quad I(f_0) = 2\mu_0 \left(\log \frac{1}{\mu_0} + M_0 \right).$$

We will use the following simple (and well-known) fact:

$$I(f) = \frac{1}{2}(I(f_1) + I(f_0)) + I_1(f). \quad (4)$$

2.2 Proof overview

The inductive approach of the proof is based on the following simple observation:

Observation 2.1. *If f_1 can be $\epsilon_1\mu_1$ -approximated by a DNF of size s_1 , and if f_0 can be $\epsilon_0\mu_0$ -approximated by a DNF of size at most s_0 , then f can be $\frac{\epsilon_1\mu_1 + \epsilon_0\mu_0}{2}$ -approximated by a DNF of size at most $s_1 + s_0$.*

It follows that if ϵ_1, ϵ_2 are chosen such that $\epsilon_1\mu_1 + \epsilon_0\mu_0 = 2\epsilon\mu$, then we have

$$s_f(\epsilon) \leq s_{f_1}(\epsilon_1) + s_{f_0}(\epsilon_0). \quad (5)$$

We perform the inductive step, rearranging the coordinates such that coordinate 1 is the *most influential* one. We distinguish between three cases:

- **Both $\min\{\mu_0, \mu_1\}$ and $I_1(f)$ are ‘not too small’.** In this case, we use (5) to combine an ϵ_1 -approximation of f_1 with an ϵ_0 -approximation of f_0 into an approximation of f . We choose ϵ_1, ϵ_0 in such a way that $\frac{M_0}{\epsilon_0} = \frac{M_1}{\epsilon_1}$, so that the sizes of the DNFs approximating f_1 and f_0 will be roughly equal. While this step doubles the size of the approximating DNF (compared to those approximating f_1, f_0), we show that $\epsilon_1/M_1, \epsilon_0/M_0$ which replace ϵ/M are larger than ϵ/M by at least a fixed amount (which depends on ϵ), and so, the number of required ‘doubling’ steps will be eventually bounded.
- **$\min\{\mu_0, \mu_1\}$ is ‘small’.** Of course, we may assume w.l.o.g. that μ_0 is small. In this case, it is better to approximate f_0 by the constant 0 function, rather than waste any subcubes on it. This step does not increase the size of the DNF, but seems to make the approximation worse. We show that nevertheless, the proof can go through, exploiting the (relatively) large influence of the first coordinate.
- **$I_1(f)$ is ‘small’.** We conclude the proof by showing that this case is impossible, as any function with a small total influence must have an influential coordinate. This is the main part of the proof, encapsulated in Lemma 1.7.

3 The Central Lemmas

In this section we prove the two central lemmas needed for the proof of Theorem 1.6.

3.1 Low-influence functions have an influential coordinate

In this subsection we prove Lemma 1.7. The proof requires two different types of tools – Fourier-theoretic and combinatorial.

The Fourier-theoretic tool we use is the classical KKL theorem [20]. (The version presented here is taken from Section 9.6 of [26], where it is called ‘the KKL edge isoperimetric theorem’).

Theorem 3.1 (Kahn, Kalai, and Linial). *Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a non-constant Boolean function, and let $\tilde{I}(f) = \frac{I(f)}{4\mathbb{E}[f](1-\mathbb{E}[f])}$. Then*

$$\max_{1 \leq i \leq n} I_i(f) \geq \frac{9}{\tilde{I}(f)^2} 9^{-\tilde{I}(f)}.$$

The combinatorial tool is the classical *shifting operators* \mathcal{S}_{ST} , introduced by Erdős, Ko, and Rado [10] and developed by Daykin [6] and others.

For $\mathbf{x} \in \{0, 1\}^n$ and $S \subset [n]$, we write $\mathbf{x}_S = 1$ if $x_i = 1$ for all $i \in S$. Similarly, we write $\mathbf{x}_S = 0$ if $x_i = 0$ for all $i \in S$. We also write $1_S \in \{0, 1\}^n$ for the indicator vector of S (i.e., $1_S(i) = 1$ if and only if $i \in S$).

Definition 3.2. Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function, and let $S, T \subseteq [n]$ be disjoint sets. The ‘shifted function’ $\mathcal{S}_{ST}(f)$ is defined by setting

$$\mathcal{S}_{ST}(f)(\mathbf{x}) := \begin{cases} f(\mathbf{x}) \wedge f(\mathbf{x} \oplus 1_{S \cup T}) & \text{if } \mathbf{x}_S = 1 \text{ and } \mathbf{x}_T = 0 \\ f(\mathbf{x}) \vee f(\mathbf{x} \oplus 1_{S \cup T}) & \text{if } \mathbf{x}_T = 1 \text{ and } \mathbf{x}_S = 0 \\ f(\mathbf{x}) & \text{otherwise.} \end{cases}$$

A more intuitive definition of the shifting operator \mathcal{S}_{ST} is as follows. Write $f = 1_A$ for $A \subset \{0, 1\}^n$. The operator \mathcal{S}_{ST} takes all elements $\mathbf{x} \in A$ such that $\mathbf{x}_S = 1$, $\mathbf{x}_T = 0$, and $\mathbf{x} \oplus 1_{S \cup T} \notin A$, and replaces them with $\mathbf{x} \oplus 1_{S \cup T}$. All other elements of A are left unchanged.

The shifting operators will be useful for us due to the following well-known Lemma.

Lemma 3.3. Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function of measure $\mu(f) \leq \frac{1}{2}$. Write

$$f^0 = \mathcal{S}_{\emptyset\{1\}} \circ \mathcal{S}_{\emptyset\{2\}} \circ \cdots \circ \mathcal{S}_{\emptyset\{n\}}(f),$$

$$f^1 = \mathcal{S}_{\{n\}\{1\}} \circ \mathcal{S}_{\{n-1\}\{1\}} \circ \cdots \circ \mathcal{S}_{\{2\}\{1\}}(f^0),$$

$$\vdots$$

$$f^n = \mathcal{S}_{\{n, \dots, 2\}\{1\}}(f^{n-1}).$$

Then:

- $I_i(f^n) \leq I_i(f)$ for any $i \geq 2$.
- $I(f^n) \leq I(f)$.
- The function f^n satisfies $f^n(0, x_2, x_3, \dots, x_n) = 0$ for all x_2, \dots, x_n .

Now we are ready to present the proof of Lemma 1.7. For convenience, we recall the statement of the Lemma.

Lemma 1.7. There exists an absolute constant C_1 such that the following holds. Let $M, \delta > 0$ satisfy $M/\delta > C$, and let $\mu \in (0, 1 - \delta)$. Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function with $\mathbb{E}[f] = \mu$, and suppose that $I(f) \leq 2\mu(\log(1/\mu) + M)$. Then

$$\max_{i \in [n]} \{I_i(f)\} \geq 2^{-C_1 M/\delta} \mu.$$

Proof. Suppose first that $\mu \geq \frac{1}{4}$. In this case, we have

$$\tilde{I}(f) = \frac{I(f)}{4\mathbb{E}[f](1 - \mathbb{E}[f])} \leq \frac{4}{3} \cdot (M/\delta),$$

and thus, the assertion follows immediately from Theorem 3.1.

Now suppose that $\mu \leq \frac{1}{4}$. Let $i \in \mathbb{N}$ be such that $\mu \in [2^{-1-i}, 2^{-i})$. The proof will proceed by induction on i . Let f^n be as in Lemma 3.3, and define $f_1^n, f_0^n: \{0, 1\}^{n-1} \rightarrow \{0, 1\}$ by $f_1^n(\mathbf{x}) = f^n(1, \mathbf{x})$ and $f_0^n(\mathbf{x}) = f^n(0, \mathbf{x})$. Recall that by Lemma 3.3, we have $f_0^n(\mathbf{x}) \equiv 0$. Thus,

$$2\mu(\log(1/\mu) + M) \geq I(f^n) = \frac{1}{2}I(f_1^n) + \frac{1}{2}I(f_0^n) + I_1(f^n) = 2\mu + \frac{1}{2}I(f_1^n),$$

where the leftmost equality follows from (4). Write $\mu_1 = 2\mu = \mu(f_1^n)$. We obtain

$$I(f_1^n) \leq 2\mu_1(\log(1/\mu_1) + M).$$

By the induction hypothesis, the maximal influence of f_1^n is at least $2^{-C_1 M/\delta} \mu_1$. This implies that $I_i(f^n) \geq 2^{-C_1 M/\delta} \frac{\mu_1}{2}$ for some $i \geq 2$. By Lemma 3.3, it follows that $I_i(f) \geq 2^{-C_1 M/\delta} \mu$. This completes the proof. \square

3.2 The effect of an influential coordinate on the restricted functions in the induction process

In this subsection we suppose w.l.o.g. that $I_1(f)$ is the maximal influence of f . By Lemma 1.7, $I_1(f)$ is ‘not very small’. We show that in this case, when we perform the induction process on the first coordinate (as described in Section 2), the influences $I(f_1)$ and $I(f_0)$ are, on average, ‘closer to the minimum’ than $I(f)$. On the intuitive level, this is apparent in view of (4), but we need a quantitative result. The ‘advantage’ we obtain here will be crucial in the inductive step of Theorem 1.6, both in the case where μ_0 is small (where it will compensate for a looser approximation, resulting from approximating f_0 by the zero function), and in the case where μ_0, μ_1 , and $I_1(f)$ are all large (where it will allow to bound the number of steps that double the size of the approximating DNF).

The following lemma was proved by Ellis [8].

Lemma 3.4. *There exists an absolute constant c such that the following holds. Let $\zeta > 0$ and let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. If $\min(I_1(f), \mu_0, \mu_1) \geq \zeta\mu$, then*

$$2M\mu - M_1\mu_1 - M_0\mu_0 \geq c\zeta\mu.$$

We prove a similar result in the case where μ_0 (or, equivalently, μ_1) is small.

Lemma 3.5. *Let $C_3 > 0$. Suppose that*

$$\min\{\mu_0, \mu_1\} \leq 2^{-C_3}\mu.$$

Then

$$2M\mu - M_1\mu_1 - M_0\mu_0 \geq (C_3 - 1) \min(\mu_0, \mu_1).$$

Proof. We assume w.l.o.g. that $\mu_0 \leq \mu_1$. The lemma follows from a straightforward computa-

tion:

$$\begin{aligned}
2M\mu - M_1\mu_1 - M_0\mu_0 &= I(f) - 2\mu \log \frac{1}{\mu} - \frac{1}{2} \left(I(f_1) - 2\mu_1 \log \frac{1}{\mu_1} \right) \\
&\quad - \frac{1}{2} \left(I(f_0) - 2\mu_0 \log \frac{1}{\mu_0} \right) \\
&= I_1(f) + \mu_1 \log \frac{1}{\mu_1} + \mu_0 \log \frac{1}{\mu_0} - 2\mu \log \frac{1}{\mu} \\
&\geq \mu_1 - \mu_0 + \mu_1 \log \frac{1}{\mu_1} + \mu_0 \log \frac{1}{\mu_0} - 2\mu \log \frac{1}{\mu} \\
&= \mu_1 \log \frac{2}{\mu_1} + \mu_0 \log \frac{1}{2\mu_0} - 2\mu \log \frac{1}{\mu} \\
&\geq \mu_1 \log \frac{1}{\mu} + \mu_0 \log \frac{1}{2\mu_0} - 2\mu \log \frac{1}{\mu} \\
&= \mu_0 \left(\log \frac{1}{2\mu_0} - \log \frac{1}{\mu} \right) \\
&= \mu_0 \log \left(\frac{\mu}{2\mu_0} \right) \geq (C_3 - 1) \mu_0.
\end{aligned}$$

□

4 Proof of the Main Theorem

Definition 4.1. Let $\mu \in (0, 1)$, $\epsilon > 0$, and $n \in \mathbb{N}$. We define $\tilde{s}(\mu, \epsilon, n)$ to be the smallest integer such that the following holds. Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function, and write

$$I(f) = 2\mu \left(\log \left(\frac{1}{\mu} \right) + M \right).$$

Then f can be $\epsilon M \mu$ -approximated by a DNF of size $\tilde{s}(\mu, \epsilon, n)$.

We also write $\tilde{s}(\epsilon)$ for the supremum of $\tilde{s}(\mu, \epsilon, n)$ over all $\mu \in (0, 1)$, and all $n \in \mathbb{N}$.

It is clear that in order to prove Theorem 1.6, it is sufficient to show that

$$\tilde{s}(\epsilon) \leq 2^{2^{O(\frac{1}{\epsilon})}} \quad (6)$$

for any $\epsilon > 0$. Throughout this section, we assume w.l.o.g. that $I_1(f)$ is the maximal influence of f , and that $\mu_0 \leq \mu_1$.

First, we show that one can assume w.l.o.g. that $\epsilon < C_4$ for a constant C_4 . This follows immediately from the stability version of Theorem 1.1 proved by the first author [8].

Theorem 4.2 ([8]). *There exist an absolute constant $c' > 0$ such that the following holds. Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function with $\mathbb{E}[f] = \mu$, and let $\epsilon > 0$. Suppose that*

$$I(f) \leq \mu (\log(1/\mu) + c' \epsilon \log(1/\epsilon)).$$

Then f can be $\epsilon \mu$ -approximated by a subcube.

Lemma 4.3. *There exists an absolute constant C_4 such that for all $\epsilon > C_4$,*

$$\tilde{s}(\epsilon) = 1.$$

Proof. Let $\epsilon > C_4$ for C_4 to be specified below, and let $f: \{0,1\}^n \rightarrow \{0,1\}$ be a Boolean function. Write $I(f) = 2\mu \left(\log \frac{1}{\mu} + M \right)$. We have to show that f can be $\epsilon M \mu$ -approximated by a subcube. If $M\epsilon \geq 1$, then f can be approximated by the constant 0 function. Thus, we may assume that $M \leq \frac{1}{C_4} \leq 1$, provided that $C_4 \geq 1$. By Theorem 4.2, there exists $c' > 0$, such that f can be $c' \frac{M}{\log(1/M)} \mu$ -approximated by a subcube. Hence, f can be $\epsilon M \mu$ -approximated by a subcube provided that C_4 is sufficiently large. This completes the proof. \square

Now we present the main part of the inductive argument. We show that there exists $C_5 > 0$ such that in any step of the inductive process, one of the following alternatives must occur:

1. Either there exists some $\mu_1 \geq \mu$, such that

$$\tilde{s}(\mu, \epsilon, n) \leq \tilde{s}(\mu_1, \epsilon, n-1),$$

2. Or

$$\tilde{s}(\mu, \epsilon, n) \leq 2\tilde{s}(\epsilon + 2^{-C_5/\epsilon}).$$

This will follow immediately from combination of two claims:

Claim 4.4. *There exists $C_6 > 0$ such that the following holds. If $\mu_0 \leq 2^{-C_6/\epsilon} \mu$, then f can be $\epsilon M \mu$ -approximated by a DNF of size at most $\tilde{s}(\mu_1, \epsilon, n-1)$.*

Claim 4.5. *Let $C_6 > 0$ be some constant, and suppose that $\mu_0 \geq 2^{-C_6/\epsilon} \mu$. Then f can be $\epsilon M \mu$ -approximated by a DNF of size at most $2\tilde{s}(\epsilon + 2^{-C_5/\epsilon})$, provided that C_5 is large enough.*

Proof of Claim 4.4. Note that f_1 can be $\epsilon M_1 \mu_1$ -approximated by a DNF of size $s' := s(\epsilon, \mu_1, n)$, say $T_1 \vee T_2 \vee \dots \vee T_{s'}$. This implies that f can be $(\frac{1}{2}\epsilon M_1 \mu_1 + \frac{1}{2}\mu_0)$ -approximated by the DNF $(1 \wedge T_1) \vee (1 \wedge T_2) \vee \dots \vee (1 \wedge T_{s'})$. The claim will follow once we show that $\frac{1}{2}\epsilon M_1 \mu_1 + \frac{1}{2}\mu_0 \leq \epsilon M \mu$, provided that C_6 is large enough.

We may assume that $M \leq \frac{1}{\epsilon}$, for otherwise f is $\epsilon M \mu$ -approximated by the constant 0 function. By Lemma 4.3, there exists an absolute constant C_4 , such that $\tilde{s}(\mu, \epsilon, n) = 1$ provided that $\epsilon \geq C_4$. Thus, we may assume that $\epsilon \leq C_4$. By Lemma 3.5,

$$2M\mu - M_1\mu_1 - M_0\mu_0 \geq (C_6/\epsilon - 1)\mu_0 \geq \left(M + \frac{C_6 - 1 - C_4}{\epsilon}\right)\mu_0 \geq (M + 2/\epsilon)\mu_0, \quad (7)$$

where the last inequality holds provided that C_6 is large enough. Substituting $\mu = \frac{\mu_1 + \mu_0}{2}$ in (7), we obtain

$$(M - M_1)\mu_1 + M\mu_0 \geq (M - M_1)\mu_1 + (M - M_0)\mu_0 \geq \left(\frac{2}{\epsilon} + M\right)\mu_0.$$

Rearranging yields

$$M_1\mu_1 + \frac{2\mu_0}{\epsilon} \leq \mu_1 M \leq 2M\mu. \quad (8)$$

We now multiply (8) by $\frac{\epsilon}{2}$ to finish the proof of the claim. \square

Proof of Claim 4.5. As mentioned before, we may assume that $M \leq \frac{1}{\epsilon}$. By Lemma 1.7, there exists $C_1 > 0$, such that $I_1(f) \geq 2^{-C_1/\epsilon} \mu$. By Lemma 3.4, there exists $c > 0$, such that

$$2M\mu - M_1\mu_1 - M_0\mu_0 \geq c2^{-\max\{C_1, C_6\}/\epsilon} \mu. \quad (9)$$

Write $B = c2^{-\max\{C_1, C_6\}/\epsilon}$ and $\epsilon' = \frac{2M}{2M-B}\epsilon$. Let D_1 be the DNF of size at most $\tilde{s}(\mu_1, \epsilon', n-1)$ that $\epsilon'M_1\mu_1$ -approximates f_1 , and let D_0 be the DNF of size at most $\tilde{s}(\mu_0, \epsilon', n-1)$ that $\epsilon'M_0\mu_0$ -approximates f_0 . Let $(x_1 \wedge D_1) \vee (\neg x_1 \wedge D_0)$ be the DNF defined by adding the literal x_1 to each term of D_1 , adding the literal $\neg x_1$ to each term of D_0 , and conjuncting the resulting DNFs. The size of the resulting DNF is at most $\tilde{s}(\mu_1, \epsilon', n-1) + \tilde{s}(\mu_0, \epsilon', n-1) \leq 2\tilde{s}(\epsilon')$, and it clearly $\frac{1}{2}\epsilon'M_0\mu_0 + \frac{1}{2}\epsilon'M_1\mu_1$ -approximates f . By (9) we have

$$\frac{1}{2}\epsilon'M_0\mu_0 + \frac{1}{2}\epsilon'M_1\mu_1 \leq \epsilon M\mu,$$

and thus, f can be $\epsilon M\mu$ -approximated by a DNF of size at most $2\tilde{s}(\epsilon')$. Finally, provided that C_5 is large enough, we have $\epsilon' = \frac{2M}{2M-B}\epsilon \geq 2^{-C_5/\epsilon} + \epsilon$. This completes the proof. \square

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. By Lemma 4.3, there exists an absolute constant C_4 such that $\tilde{s}(\epsilon) = 1$ for any $\epsilon \geq C_4$. Let C_5 be the constant from Claim 4.5. By combination of Claims 4.4 and 4.5, a simple inductive argument implies that $\tilde{s}(\epsilon) \leq 2\tilde{s}(\epsilon + 2^{-C_5/\epsilon})$. Applying this inequality repeatedly $C_4 2^{C_5/\epsilon}$ times, we obtain $\tilde{s}(\epsilon) \leq C_4 2^{2^{C_5/\epsilon}}$. This completes the proof. \square

5 Sharpness Example

In this section we present in detail the sharpness example for Theorem 1.6, that is also a counterexample to Conjecture 1.4.

The example is based on the classical ‘tribes’ function that was introduced by Ben-Or and Linial [3] in 1985 and is known to be an extremal example for numerous results on Boolean functions.

Definition 5.1. *The tribes function of width w and size s is defined by*

$$\text{ Tribes}_{w,s}(\mathbf{x}) = (x_1 \wedge x_2 \wedge \cdots \wedge x_w) \vee (x_{w+1}, \dots, x_{2w}) \vee \cdots \vee (x_{(s-1)w+1} \wedge \cdots \wedge x_{sw}).$$

The dual of the tribes function is the function $\text{ Tribes}_{w,s}^\dagger$ defined by

$$\text{ Tribes}_{w,s}^\dagger(\mathbf{x}) = 1 - \text{ Tribes}_{w,s}(\bar{\mathbf{x}}),$$

where $\bar{\mathbf{x}}$ is the vector obtained from \mathbf{x} by flipping all of its coordinates.

We will use two well-known results: one regarding properties of the dual tribes function, and another regarding approximation by DNFs.

Theorem 5.2 ([28]). *Let $w \in \mathbb{N}$, and let $f = \text{ Tribes}_{w,2^w}^\dagger(\mathbf{x})$. Then $I(f) = \Theta(w)$, and f cannot be 0.2-approximated by a DNF of width at most $\frac{1}{3}2^w$.*

Lemma 5.3. *Let D be a DNF of size s . Then it can be $s2^{-w}$ -approximated by a DNF of width at most w and of size at most s .*

Proof. Remove from D all terms that contain more than w literals to obtain a new DNF, D' . A union bound implies that D' $s2^{-w}$ -approximates D . This completes the proof. \square

Now we are ready to present our tightness example for Theorem 1.6.

Proposition 5.4. *Let $w, l \in \mathbb{N}$, let $n = w2^w + l$, let f be the function*

$$f(\mathbf{x}) = \begin{cases} \text{Tribes}_{w,2^w}^\dagger(x_1, x_2, \dots, x_n - l) & x_{n-l+1} = \dots = x_n = 1 \\ 0 & \text{Otherwise} \end{cases},$$

and write $\mu = \mathbb{E}[f]$. Then on the one hand, $I(f) = 2\mu(\log(1/\mu) + \Theta(w))$. On the other hand, f cannot be 0.2μ -approximated by any DNF of width at most $\log(1/\mu) + \Theta(2^w)$. As a consequence, f cannot be 0.1μ -approximated by a DNF of size at most $2^{\Theta(2^w)}$.

Proof. Suppose that D is a DNF that 0.2μ -approximates f . Without loss of generality, we may assume that all the terms of D contain the variables x_{n-l+1}, \dots, x_n . Let D' be the DNF obtained from D by removing the variables x_{n-l+1}, \dots, x_n from all its terms. Then the DNF D' is $(0.2 \cdot 2^l \cdot \mu(f))$ -approximated by the function $\text{Tribes}_{w,2^w}^\dagger$. Theorem 5.2 implies that the width of D' is at least $\Theta(w)$. This completes the proof of the first part of the corollary. The “as a consequence” statement follows immediately from Lemma 5.3. This completes the proof. \square

6 Open Problems

We conclude this paper with a few open problems.

Functions with influence within a constant multiplicative factor from the minimum possible. While Theorem 1.6 describes rather precisely the structure of functions with $I(f) \leq 2\mu(f)(\log(1/\mu(f)) + o(\log(1/\mu(f))))$, the result we obtain for $I(f) = c\mu(f)\log(1/\mu(f))$ is not stronger than what one can get from Friedgut’s Junta theorem. In [19], Kahn and Kalai presented several conjectures on the structure of such functions (one of them is Conjecture 1.4 above), and it will be interesting to see whether our techniques can be helpful in addressing them.

Biased functions with respect to a biased measure. As described in the introduction, structure theorems for balanced functions with respect to a biased measure μ_p on the discrete cube were studied in numerous papers (e.g., [4, 5, 12, 18]). Our paper deals with biased functions with respect to the uniform measure. Hence, the next natural goal in this respect is to study biased functions with respect to a biased measure.

To this end, one may use the classical techniques for reduction from the biased measure to the uniform measure (see, e.g., [13, 23]) to obtain a biased-measure version of Theorem 1.6. However, this version holds only when both p and μ are not very small. It seems that more powerful techniques will be needed to address the (biased function, biased measure) case.

A sharper approximation by a Junta? We tend to believe that Theorem 1.6 can be strengthened into an improved ‘approximation by Junta’ theorem. Specifically, the following conjecture seems reasonable:

Conjecture 6.1. *For any $M, \epsilon > 0$, any function $f: \{0, 1\}^n \rightarrow \{0, 1\}$ that satisfies $I(f) \leq 2\mu(f)(\log(1/\mu) + M)$ can be $\epsilon\mu$ -approximated by a function g that depends on at most $O(\log(1/\mu) \cdot 2^{M/\epsilon})$ coordinates.*

For $I(f) = c\mu(f)\log(1/\mu(f))$, Conjecture 6.1 is no better than the Junta theorem, but in the range $I(f) = 2\mu(f)(\log(1/\mu(f)) + M)$ with $M = o(\log(1/\mu(f)))$, the size of Junta it yields is much smaller. In particular, when M is constant, it becomes as small as the clearly optimal $\Theta(\log(1/\mu))$.

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